RAPID COMMUNICATIONS

PHYSICAL REVIEW E VOLUME 47, NUMBER 2 FEBRUARY 1993

Controlling chaos in highly dissipative systems: A simple recursive algorithm

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(Received 21 October 1992)

We present a recursive proportional-feedback (RPF) algorithm for controlling deterministic chaos. The algorithm is an adaptation of the method of Dressler and Nitsche [Phys. Rev. Lett. **68**, 1 (1992)] to highly dissipative systems with a dynamics that shows a nearly one-dimensional return map of a single variable X measured at each Poincaré cycle. The result extends the usefulness of simple proportional-feedback control algorithms. The change in control parameter prescribed for the nth Poincaré cycle by the RPF algorithm is given by $\delta p_n = K(X_n - X_F) + R\delta p_{n-1}$, where X_F is the fixed point of the target orbit, and K and R are proportionality constants. The recursive term is shown to arise fundamentally because, in general, the Poincaré section of the attractor near X_F will change position in phase space as small changes are made in the control parameter. We show how to obtain K and R from simple measurements of the return map without any prior knowledge of the system dynamics and report the successful application of the RPF algorithm to model systems from chemistry and biology where the recursive term is necessary to achieve control.

PACS number(s): 05.45.+b, 06.70.Td, 87.10.+e

I. INTRODUCTION

A control algorithm for stabilizing unstable saddle orbits that densely populate a chaotic attractor by applying small changes to a single control parameter was first proposed by Ott, Grebogi, and Yorke [1] (OGY). The general OGY algorithm is dramatically simplified in highly dissipative systems that are well characterized by a one-dimensional return map. Peng, Petrov, and Showalter [2], using model systems, and Hunt [3], by experimentally controlling the diode resonator circuit, demonstrated that control can be achieved by applying a δp_n on the nth Poincaré cycle that is simply proportional to the deviation of X_n from the fixed point X_F of the target orbit on the return map. In this paper we refer to this type of control algorithm as simple proportional feedback (SPF). Many nonlinear chemical and biological systems tend to be highly dissipative and SPF would seem to be applicable.

However, Petrov, Peng, and Showalter [4] recently demonstrated that the SPF algorithm does not always work for some choices of the control parameter even when the system is shown to have one-dimensional map behavior. If the Poincaré section of the attractor moves in phase space when the parameter p is changed and if $p_{n-1} \neq p_n$, then X_n is *not on* the attractor corresponding to $p = p_n$ and the usual static one-dimensional map is not valid. Petrov, Peng, and Showalter suggest a clever modification of SPF in such a case. They point out that, at least in some cases, a linear combination of control parameters can be found that does not move the Poincaré section. The linear combination can then be used as the single control parameter with the SPF algorithm to affect control. But, in a real experimental situation, it may not be possible to get experimental access to a particular control parameter, and the required linear combination of parameters needed to save the SPF method may be impossible to realize.

Dressler and Nitsche [5] recently showed that a similar situation exists when time-delay coordinates are used to reconstruct the phase space in which the OGY algorithm is applied. They show that the general OGY algorithm must be modified such that the correction δp_n applied during the current cycle depends not only on the vector deviation of the system from the fixed point but also on the correction δp_{n-1} made during the previous cycle.

In this paper we apply the Dressler and Nitsche [5] approach to the case where the system dynamics is reconstructed from measurements of a one-dimensional return map. The result is a recursive proportional-feedback (RPF) algorithm that allows control with any choice of a single control parameter. In the following sections we derive the RPF algorithm, show how the proportionality constants necessary for implementation of the algorithm are determined, and apply the algorithm to models of both chemical and biological systems where the recursive term is necessary.

II. DEVELOPMENT OF RPF ALGORITHM

The one-dimensional return map is the result of high dissipation, which causes the reduction of the chaotic attractor to a very thin (nearly two-dimensional) structure embedded in a three-dimensional phase space (X,Y,Z) of the system. Neglecting the thickness of the attractor, a Poincaré section through the two-dimensional attractor at $Z = Z_c$ gives a one-dimensional (1D) curve. (Note that Z could be dX/dt obtained from measurements of a time series of the single variable X so that the Poincaré section, and one-dimensional map, can be experimentally obtained from measurements of a single variable.) For a particular value of the control param-

eter p, the one-dimensional curve may be expressed as $Y = h(X;p)$, and the Poincaré map equation for X is of the form $X_{n+1} = g(X_n, Y_n; p) = g(X_n, h(X_n;p);p)$, or $X_{n+1} = f(X_n;p)$. This is the usual 1D return map that depends parametrically on the single parameter p.

Now we consider the case where p is changed by a small amount δp about $p = p_0$ on each Poincaré cycle. If we change from p_{n-1} to p_n at the beginning of the nth cycle, then the system will be on the attractor corresponding to p_{n-1} at the start of the cycle. We assume the high dissipation ensures that the system will settle onto the attractor corresponding to p_n by the end of the Poincaré cycle. Thus,

$$Y_n = h(X_n; p_{n-1}) \tag{1}$$

and at the *end* of the nth cycle $X = X_{n+1} = g(X_n, Y_n; p_n) = g(X_n, h(X_n; p_{n-1}); p_n)$. Thus, the general 1D return-map equation is

$$X_{n+1} = f(X_n; p_{n-1}, p_n). \tag{2}$$

Note that if Eq. (1) does not depend on p_{n-1} (i.e., the Poincaré section of the attractor does not move in phase space as p is changed), then the simple map of the previous paragraph is recovered. Figures 1(a) and 1(b) show how small changes in two different model parameters affect the Poincaré section of a chaotic attractor for a model describing metal passivation in an electrochemical cell discussed in Sec. IV. If model parameter s is changed slightly, then Fig. 1(a) shows that the attractor moves such that its Poincaré section does not shift position. However, if model parameter p is changed, the Poincaré section moves as shown in Fig. 1(b). The SPF algorithm fails with p as the control parameter.

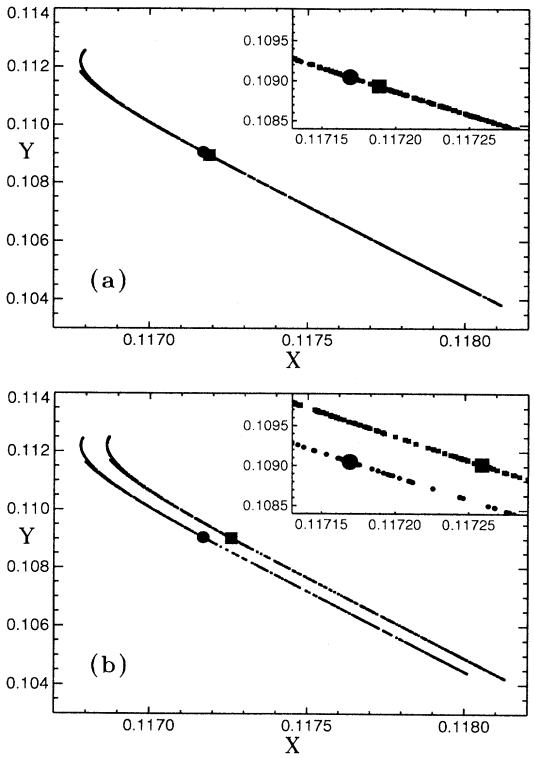

FIG. 1. Poincaré sections for the chemical model discussed in Sec. IV. (a) Circles, for model parameter $s = s_0 = 9.7 \times 10^{-5}$; and squares, for $s = s_0 + \delta s$, where $\delta s = 7 \times 10^{-8}$. (b) Circles, for model parameter $p = p_0 = 2 \times 10^{-4}$; and squares, for $p = p_0 + \delta p$, where $\delta p = 7 \times 10^{-8}$. The enlarged symbol is the position of the fixed point for the period-one orbit in each case.

Now we consider the natural dynamics of the system described by Eq. (2) near the fixed point of the period-one orbit for $p = p_0$; $X_F = f(X_F, p_0, p_0)$. To first order in $\delta X_n = (X_n - X_F)$, δp_{n-1}, and δp_n, we have

$$\delta X_{n+1} = \mu \delta X_n + w \delta p_{n-1} + v \delta p_n, \tag{3}$$

where $\mu = \partial f(X_F, p_0, p_0)/\partial X_n$, $w = (\partial g/\partial Y_n)(\partial h/\partial p_{n-1})$, and $v = \partial f(X_F, p_0, p_0)/\partial p_n$. Note that μ is the slope of the one-dimensional return map at the fixed point for $p = p_0$ and $w = 0$ if h is independent of p.

We want to determine the δp_n such that the system is brought to the fixed point as quickly as possible. Several strategies could be applied. We use the strategy $\delta X_{n+2} = 0$ and $\delta p_{n+1} = 0$ and then the first and second iterate of Eq. (3) give the recursive control algorithm:

$$\delta p_n = K \delta X_n + R \delta p_{n-1}, \tag{4}$$

where

$$K = -\frac{\mu^2}{(\mu v + w)}, \text{ and } R = -\frac{\mu w}{(\mu v + w)}. \tag{5}$$

III. DETERMINATION OF PARAMETERS

The parameters necessary to implement the RPF method (X_F, μ, w, and v) can be easily determined from two types of 1D return-map measurements. For a constant parameter value near p_0, a plot of X_{n+1} versus X_n defines the 1D mapping function $f_c(X;\delta p) \equiv f(X; p_0 + \delta p, p_0 + \delta p)$. The value of $X_F = f_c(X_F; 0)$ and the slope μ of the function f_c at $X = X_F$ are obtained from X_n data taken with $\delta p = 0$. The values of v and w are obtained from the sequence X_n's measured while repeatedly alternating the control parameter *up* to $p_0 + \delta p$ for one Poincaré cycle and then *back* to p_0 for the next cycle. Alternate pairs (X_n,X_{n+1}) taken from this measured sequence lie on two maps $f_u(X;\delta p)$ and $f_b(X;\delta p)$ where $f_u(X_n;\delta p) \equiv f(X_n; p_0, p_0 + \delta p) = f_c(X_n;0) + v\delta p$ and $f_b(X_n;\delta p) \equiv f(X_n; p_0 + \delta p, p_0) = f_c(X_n;0) + w\delta p$. An example of the three return maps is shown in Fig. 2. It is easiest to measure the fixed points of these two maps, $X_F^u = f_u(X_F^u;\delta p)$ and $X_F^b = f_b(X_F^b;\delta p)$. We define g_u and g_b such that $X_F^u = X_F + g_u \delta p$ and $X_F^b = X_F + g_b \delta p$. Then, as shown in Fig. 2, w and v can be expressed in terms of μ, g_u, and g_b by $w = (1-\mu)g_u$, and $v = (1-\mu)g_b$. Now, Eq. (5) becomes

$$K = \frac{\mu^2}{(\mu - 1)(\mu g_u + g_b)}, \quad R = -\frac{\mu g_b}{(\mu g_u + g_b)}. \tag{6}$$

Following OGY, we define the shift in the fixed point for constant shift δp in control parameter as $g\delta p$. Then, from the above definitions, we have the following sum rule:

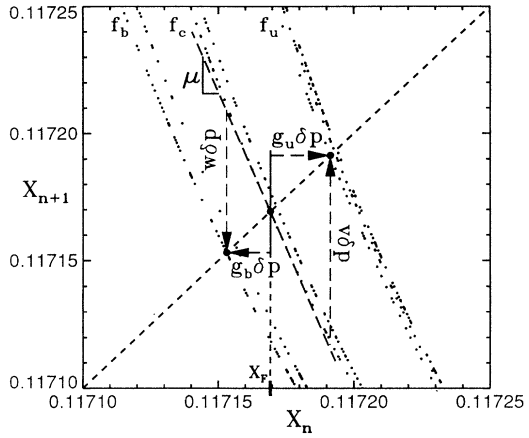

FIG. 2. Diagram showing the three (nearly 1D) mapping functions defined in the text $f_c(x;0)$, $f_u(x;\delta p)$, and $f_b(x;\delta p)$ in the neighborhood of the fixed point X_F. The data are taken from the numerical solution of the chemical model discussed in Sec. IV. The fractal nature of the return map is clear at this scale and we choose corresponding fixed points on each map. The geometrical relationships between the parameters w and v and the parameters related to the shift in the fixed points g_u and g_b are shown.

$$g = \frac{w+v}{(1-\mu)} = g_u + g_b. \tag{7}$$

Also, note that if $w = 0$ then $g_b = 0$, $g_u = g$, and $R = 0$, so that the RPF algorithm reduces to the SPF algorithm $\delta p_n = [\mu/(\mu-1)g]\delta X_n$ of Peng, Petrov, and Showalter [2].

IV. APPLICATION TO MODELS

We now apply the RPF control algorithm to stabilizing periodic orbits within a chaotic attractor in two model systems where SPF fails. The first system is a kinetic rate-equation model for metal passivation [6] and the second system is a diffusively coupled two-oscillator model [4, 10, 11] for the respiratory behavior of a bacterial culture. Both cases lead to nearly 1D return maps but SPF fails for several control parameters. A standard fourth-order Runge-Kutta algorithm was used to integrate the model equations.

In preparation for performing experimental control of chaotic behavior in a chemical system, we attempted control by computer simulation using a mathematical model of a similar system. We found that the SPF algorithm failed in controlling chaos in the model and, out of necessity, the RPF algorithm described in the previous sections was developed and successfully applied. The chemical system under consideration is the passivation of the reactive surface of a metal electrode in an electrochemical cell. The chemical kinetics of the passivation model includes the formation of two surface films, MOH and MO, where M represents the metal atom. It combines elements from surface reaction models by Talbot and Oriani [7] for MOH and by Sato [8] for MO formation. The chemical kinetics lead to the dimensionless equations [9]

$$\dot{Y} = p(1-\theta_{OH}-\theta_O) - qY, \tag{8}$$

$$\dot{\theta}_{OH} = Y(1-\theta_{OH}-\theta_O) - [\exp(-\beta\theta_{OH}) + r]\theta_{OH} + 2s\theta_O(1-\theta_{OH}-\theta_O), \tag{9}$$

$$\dot{\theta}_O = r\theta_{OH} - s\theta_O(1-\theta_{OH}-\theta_O), \tag{10}$$

where Y is the concentration of metal ions in the electrolyte, θ_{OH} and θ_O are the respective fraction of the metal surface covered by each film, p, q, r, and s are parameters related to chemical rate constants, and β represents the non-Langmuir nature of MOH film formation in the Talbot-Oriani model. The system has been studied recently in some detail [9] and is chaotic for parameter values $(p, q, r, s, \beta) = (2.0\times10^{-4}, 1.0\times10^{-3}, 2.0\times10^{-5}, 9.7\times10^{-5}, 5.0)$. The discussion below concerns the behavior of the system in the neighborhood of this point in parameter space.

In an experimental cell, the parameter p is related to the anodic potential, which can be changed by the experimenter. The parameter s, on the other hand, is related to rate constants that are not easily accessible to the experimenter. Figures 1(a) and 1(b) show that the Poincaré section moves when the experimentally accessible parameter p is changed (the variable X in this case is the metal-oxide film coverage θ_O and the Y coordinate is proportional to the concentration of metal ions in solution). Attempts to control using the SPF algorithm were successful when using s but failed when using the experimentally accessible parameter p. The RPF algorithm was applied to stabilize the period-one unstable saddle orbit. We used $\delta p = 5\times10^{-9}$ when alternating p to obtain $f_u(X;\delta p)$ and $f_b(X;\delta p)$. Figure 2 shows the three return maps $f_c(\,)$, $f_u(\,)$, and $f_b(\,)$ for $X \equiv \theta_O$ taken when θ_{OH} passes the Poincaré plane $\theta_{OH} = 0.3125$ with $\dot{\theta}_{OH} < 0$. The values $X_F = 0.117\,169\,2$, $\mu = -2.44$, $g_u = 4420$, and $g_b = -3230$ were obtained from data as shown in Fig. 2. Equation (6) then gives $K = 1.24\times10^{-4}$ and $R = 0.562$ for the RPF algorithm. While the system was on the chaotic attractor the control was turned on and the RPF correction was applied whenever $|X_n - X_F| < 4\times10^{-5}$. The results of turning the control on and off are indicated by the time series of θ_O at the Poincaré section shown in Fig. 3. We never observed the system to fall off the stabilized period-one orbit so long as the RPF algorithm was continually applied. We also used the RPF algorithm to stabilize a period-two orbit by using the second iterate of the Poincaré return map. The properties of the second iterate 1D maps gave $X_F = 0.117\,497\,88$, $\mu = -4.75$, $g_u = 3870$, and $g_b = -3259$. This gave $K = 1.18\times10^{-4}$ and $R = 0.715$. The results of turning the control on are also shown in Fig. 3.

Finally, we report the successful application of the RPF algorithm to a system for which the SPF was already shown to fail (if a single control parameter was used) by Petrov, Peng, and Showalter [4]. The respiratory behavior of bacterial culture has been modeled by the dimensionless equations [4, 11]

$$\dot{x}_1 = b_1 - x_1 - x_1y_1/(1+q_1x_1^2) + D_xc(x_2-x_1), \tag{11}$$

$$\dot{y}_1 = a_1 - x_1y_1/(1+q_1x_1^2) + D_yc(y_2-y_1), \tag{12}$$

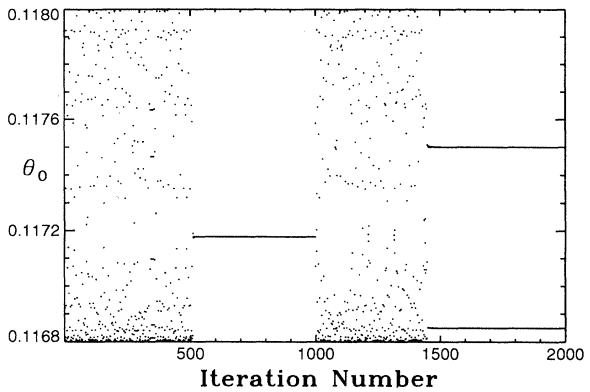

FIG. 3. Sequence of θ_O values taken at the Poincaré section (described for the chemical system in Sec. IV) as the RPF control for period-one orbit is turned on and off and then RPF for control of period-two orbit is turned on.

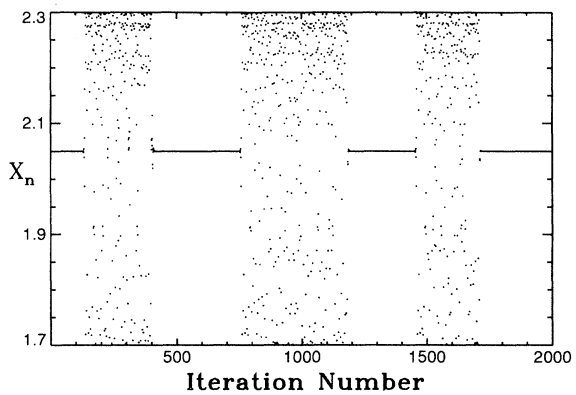

FIG. 4. Sequence of x_1 values taken at the Poincaré section (described for the biological system in Sec. IV) as RPF control is turned on and off several times.

$$\dot{x}_2 = b_2 - x_2 - x_2 y_2/(1 + q_2 x_2^2) - D_x c(x_2 - x_1), \quad (13)$$

$$\dot{y}_2 = a_2 - x_2 y_2/(1 + q_2 x_2^2) - D_y c(y_2 - y_1), \quad (14)$$

where x and y are system variables; a, b, and q are parameters of the uncoupled oscillators; D_x, D_y are the x and y diffusion coefficients; and c is the coupling strength. Petrov, Peng, and Showalter [4] found that SPF control using a single control parameter generally was not possible. They were unable to find a parameter that did not move the Poincaré section of the attractor in phase space. They were successful if they chose a special linear combination of a_1 and b_1 such that the fixed point moved along the attractor that remained stationary in the phase space. We chose the single parameter a_1 to affect the control using the RPF algorithm. Control was applied to a period-one orbit embedded in the chaotic attractor at the same point in parameter space as used by Petrov, Peng, and Showalter [4] ($a_1 = a_2$, $b_1 = b_2$, $q_1 = q_2$, D_x, D_y, c)= (8.9, 11, 0.5, 1×10^{-5}, 1×10^{-3}, 4000). We used $\delta a_1 = 0.03$ when alternating a_1 to obtain $f_u(x; \delta a_1)$ and $f_b(x; \delta a_1)$. The return maps were taken from a series of x_{1n} taken when y_2 passes the plane $y_2 = 13.45$ with $\dot{y}_2 > 0$. The values $X_F = 2.033$, $\mu = -2.34$, $g_u = -0.2302$, and $g_b = -0.1665$ were taken from data like that shown in Fig. 2. Equation (6) then gives $K = -4.403$ and $R = -1.046$, which were used to implement the RPF algorithm. While the system was on the chaotic attractor the control was turned on and the RPF correction was then applied whenever $|x_{1n} - X_F| < 0.01$. The results of turning the control on and off are indicated by the time series of x_1 at the Poincaré section shown in Fig. 4. As with the RPF control of the chemical system, we never observed the system to fall off the stabilized periodic orbit so long as the RPF algorithm was continually applied.

V. CONCLUSION

A RPF algorithm was developed for controlling chaos in systems described by one-dimensional return maps. Previous proportional-feedback algorithms, at least in the small-perturbation regime, are generally limited to control parameters that do not affect the position of the Poincaré section of the attractor in phase space. We show that the recursive algorithm removes this limitation and yet is easy to implement in real experimental systems. We believe the simple RPF algorithm is widely applicable to systems found in physics, chemistry, and biology. Our preliminary results on applying RPF to control a real chemical system are encouraging.

ACKNOWLEDGMENTS

We have benefited greatly from discussions with Alan Markworth and Kevin McCoy of the Battelle Memorial Institute in Columbus and with Jerry Orosz, Don Weekley, Gregg Johnson, Earle Hunt, and Howard Dewald at Ohio University. This work was supported in part by a Battelle subcontract of the Electric Power Research Institute (EPRI) research project Contract No. RP2426-25 and by Ohio University Research Challenge Grant No. RC89-107.

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